Periodicity of damped harmonic oscillator affected by magnetic field in time dependent noncommutative space

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A model of two-dimensional damped harmonic oscillator affected by time dependent magnetic field in time dependent noncommutative space was studied in an earlier communication [Phys. Scr. 96 (2021) 125224]. In this article, the same model is considered to show that, for a suitable explicit choices of damping and magnetic field, it is possible to import a periodicity to the time evolution of the energy expectation value of the model system.

I. INTRODUCTION

The study of quantum harmonic oscillators with time dependent parameters is quite popular amongst theoreticians[1]. The effect of damping on the time dependent quantum oscillator has also been well investigated[2]. The system has also been studied in the non commutative framework by few studies[3] including those by us[4,5]. The effect of an external magnetic field on the damped system has been analysed by us in an earlier work[6]. We have seen through this study that in the presence of a constant, exponentially decaying, exponentially growing and rationally decaying external field, the expectation value of energy either remains constant with time or decreases or initially decreases then increases with time. So, in the present work we intend to extend our earlier mentioned study to analyse the time evolution of the energetics of the damped oscillator in the presence of a periodic external magnetic field. We intend to investigate whether the periodicity of the external field is reflected in the energetics of the system.

In this communication we set up the Hamiltonian for our time dependent damped quantum harmonic oscillator in non commutative framework in the presence of an externally applied magnetic field in Section II. In Section III we solve for the eigenstate of the Hamiltonian using Lewis Invariant technique. In Section IV we have looked into explicit solutions for our system. In Section V we have studied the energetics of the system in terms of energy expectation values. In Section VI we finally summarize our results.

II. CONSIDERING THE TWO-DIMENSIONAL HARMONIC OSCILLATOR IN THE PRESENCE OF MAGNETIC FIELD IN NONCOMMUTATIVE SPACE

The model Hamiltonian considered in our earlier communication[6] to study a two dimensional damped harmonic oscillator in the presence of external magnetic field in non commutative space reads,

$$H(t) = \frac{f(t)}{2M}\left[(P_1 - qA_1)^2 + (P_2 - qA_2)^2\right] + \frac{M\omega^2(t)}{2f(t)}({X_1}^2 + {X_2}^2) \ ; \tag{1}$$

where $f(t)$, the damping factor is formed as,

$$f(t) = e^{-\int_0^t \eta(s)ds}; \tag{2}$$

where $\eta(s)$ defines the coefficient of friction and A_i denotes the vector potential of a time varying external magnetic field $B(t)$ chosen in Coulomb gauge as,

$$A_i = -\frac{B(t)}{2}\epsilon_{ij}X^j \ . \tag{3}$$

It should also be noted that $\omega(t)$ denotes the time varying angular frequency of the oscillator and M is the constant mass of the oscillator. In order to express the Hamiltonian (Eqn.(1)) in commutative space we relate the NC coordinates (X_i, P_i) to the commutative space variables (x_i, p_i) by the standard Bopp-shift relations[7] which are mentioned below in natural unit $\hbar = 1$.

$$X_1 = x_1 - \frac{\theta(t)}{2}p_2 \ , \ X_2 = x_2 + \frac{\theta(t)}{2}p_1 \ , \tag{4}$$

$$P_1 = p_1 + \frac{\Omega(t)}{2}x_2 \ , \ P_2 = p_2 - \frac{\Omega(t)}{2}x_1 \ . \tag{5}$$

Hence the Hamiltonian in terms of commutative variables (x_i, p_i) is formed as,

$$H = \frac{a(t)}{2}({p_1}^2+{p_2}^2)+\frac{b(t)}{2}({x_1}^2+{x_2}^2)+c(t)(p_1x_2-p_2x_1) \; ; \tag{6}$$

where the time dependent coefficients $a(t), b(t)$ and $c(t)$ are found as,

$$\begin{aligned} a(t) &= \frac{f(t)}{M} + \frac{qB(t)f(t)\theta(t)}{2M} + \left[\frac{q^2B^2(t)f(t)}{16M} + \frac{M\omega^2(t)}{4f(t)}\right]\theta^2(t), \\ b(t) &= \frac{q^2B^2(t)f(t)}{4M} + \frac{M\omega^2(t)}{f(t)} + \frac{qB(t)f(t)\Omega(t)}{2M} + \frac{f(t)\Omega^2(t)}{4M}, \\ c(t) &= \frac{1}{2}\left[\frac{qB(t)f(t)}{M}\left(1+\frac{\theta(t)\Omega(t)}{4}\right) + \frac{\Omega(t)f(t)}{M}\right. \\ &\quad \left. + \left(\frac{q^2B^2(t)f(t)}{4M} + \frac{M\omega^2(t)}{f(t)}\right)\theta(t)\right] \; . \end{aligned} \tag{7}$$

III. EIGENSTATE OF THE HAMILTONIAN: LEWIS INVARIANT TECHNIQUE

As we intend to follow the approach introduced by Lewis *et.al.* [1] to solve a time dependent Hamiltonian, we find a time-dependent hermitian invariant operator $I(t)$ corresponding to the Hamiltonian $H(t)$ shown in Eqn.(6). According to that well known method of invariant , if the eigenfunctions of $I(t)$ is denoted as $|\phi\rangle$ i.e.

$$I(t)\,|\phi\rangle = \epsilon\,|\phi\rangle \; ; \tag{8}$$

where ϵ is a time independent eigenvalue of $I(t)$ corresponding to $|\phi\rangle$, we can correlate the eigenstates of $H(t)$, $|\psi\rangle$ with $|\phi\rangle$ by choosing a suitable time dependent phase factor, known as Lewis phase in literature. Hence the eigenstate of $H(t)$ i.e. $|\psi\rangle$ can be expressed in terms of $|\phi\rangle$ and the phase factor $\Theta(t)$ in the following manner.

$$|\psi\rangle = e^{i\Theta(t)}\,|\phi\rangle \; ; \tag{9}$$

where $\Theta(t)$, the Lewis phase is a real function of time and satisfies the following relation,

$$\dot{\Theta}(t) = \langle\phi|\, i\partial_t - H(t)\,|\phi\rangle \; ; \tag{10}$$

where dot denotes the first time derivative.

The time dependent Lewis invariant operator $I(t)$ corresponding to the Hamiltonian $H(t)$ satisfies the following relation,

$$\frac{dI}{dt} = \partial_t I + \frac{1}{i}[I, H] = 0. \tag{11}$$

As argued by Lewis *et. al.*[1], the hermitian invariant $I(t)$ has explicit time dependence and it is to be of the same homogeneous quadratic form in the canonical variables as the Hamiltonian. Thus, the preliminary structure of the time dependent invariant in two dimension is assumed to take the following form in natural unit $\hbar = 1$,

$$I(t) = \alpha(t)({p_1}^2+{p_2}^2)+\beta(t)({x_1}^2+{x_2}^2)+\gamma(t)(x_1p_1+p_2x_2) \; ; \tag{12}$$

where $\alpha(t), \beta(t)$ and $\gamma(t)$ possess some arbitrary functional form of time. Substituting the Eqn.(12) in the Eqn.(11) and equating the coefficients of the canonical variables, the following relations are found as,

$$\begin{aligned} &\dot{\alpha}(t) = -a(t)\gamma(t) \; , \; \dot{\beta}(t) = b(t)\gamma(t) \; , \\ &\dot{\gamma}(t) = 2\left[\, b(t)\alpha(t) - \beta(t)a(t)\,\right] \end{aligned} \tag{13}$$

Next we make a parametrization $\alpha(t) = \rho^2(t)$ and substitute it in the first and third relation in the Eqn.(13) as it enables us to express the three time dependent coefficients in terms of a single time dependent parameter $\rho(t)$. Hence the other two parameters take the form in terms of $\rho(t)$ as,

$$\gamma(t) = -\frac{2\rho\dot{\rho}}{a(t)} \; , \; \beta(t) = \frac{1}{a(t)}\left[\frac{\dot{\rho}^2}{a(t)} + \rho^2 b + \frac{\rho\ddot{\rho}}{a(t)} - \frac{\rho\dot{\rho}\dot{a}}{a^2}\right] \tag{14}$$

In order to simplify the form of β we substitute it in the second relation in Eqn.(13). This in turn leads to the following non-linear differential equation,

$$\ddot{\rho} - \frac{\dot{a}}{a}\dot{\rho} + ab\rho = \xi^2\frac{a^2}{\rho^3} \; ; \tag{15}$$

which is known as Ermakov-Pinney equation having a dissipative term[2,8,9] and ξ^2 is a constant of integration. As our Hamiltonian does not differ with the same in Dey *et al* [2] , the obtained EP equation has similar form to that established by them [2]. However, it should be mentioned that the explicit form of the time-dependent coefficients in the Hamiltonian are different due to the existence of damping and external magnetic field.
Next the EP equation provides us the simplest form of β as,

$$\beta(t) \;=\; \frac{1}{a(t)}\left[\frac{\dot{\rho}^2}{a(t)} + \frac{\xi^2 a(t)}{\rho^2}\right]. \tag{16}$$

Next, the explicit form of α, β and γ in terms of $\rho(t)$ establish the following expression for $I(t)$ as,

$$\begin{aligned} I(t) =&\rho^2({p_1}^2+{p_2}^2) + \left(\frac{\dot{\rho}^2}{a^2} + \frac{\xi^2}{\rho^2}\right)({x_1}^2+{x_2}^2) \\ &- \frac{2\rho\dot{\rho}}{a}(x_1p_1+p_2x_2). \end{aligned} \tag{17}$$

Since we intend to solve the model system in polar coordinate system, the Lewis invariant in terms of the polar coordinate variables is as follows,

$$I(t) = \frac{\xi^2}{\rho^2}r^2 + \left(\rho p_r - \frac{\dot{\rho}}{a}r\right)^2 + \left(\frac{\rho p_\theta}{r}\right)^2 - \left(\frac{\rho}{2r}\right)^2 \tag{18}$$

The above form of the invariant produces the solution of the Hamiltonian (as found from[2]) as,

$$\begin{aligned}\psi_{n,m-n}(r,\theta,t) &= e^{i\Theta_{n,m-n}(t)}\phi_{n,m-n}(r,\theta)\\ &= \lambda_n \frac{(i\rho)^m}{\sqrt{m!}} \exp\left[im\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right)dT\right]\\ &\times r^{n-m} e^{i(m-n)\theta - \frac{a(t)-i\rho\dot{\rho}}{2a(t)\rho^2}r^2}\\ &\times U\left(-m, 1-m+n, \frac{r^2}{\rho^2}\right) ; \end{aligned} \tag{19}$$

where λ_n is given by

$$\lambda_n^2 = \frac{1}{\pi n! (\rho^2)^{1+n}} ; \tag{20}$$

and n and m are non negative integers.

Here, $U\left(-m, 1-m+n, \frac{r^2}{\rho^2}\right)$ is known as Tricomi's confluent hypergeometric function[10,11] and the study by Dey *et al*[2] also reveals the form of the Lewis phase as,

$$\begin{aligned}\Theta_{n,l}(t) &= (n+l)\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right)dT\\ &= m\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right)dT ; \end{aligned} \tag{21}$$

where l is an integer such that $n + l = m \geqslant 0$.

IV. EXPLICIT SOLUTION FOR DAMPED OSCILLATOR WITH MAGNETIC FIELD IN NONCOMMUTATIVE SPACE

Here the solution of the EP equation (considering $\xi^2 = 1$) under a special physical condition is discussed and the explicit form of the above solution along with the Lewis phase will be calculated under such circumstances.

A. The exponential solution set of EP equation

The simplest kind of solution set of EP equation is found by Dey *et al* in their work[2] and those are found to be varying exponentially with respect to time. The exponentially decaying set of EP equation is given by the following relations,

$$a(t) = \sigma e^{-\Gamma t} \;,\; b(t) = \Delta e^{\Gamma t} \;,\; \rho(t) = \mu e^{-\Gamma t/2}; \tag{22}$$

where σ, Δ, μ are arbitrary constants. Substitution of the above solution in the EP equation (Eqn.(15)) generates the following constraint relation among the constants used in EP solution.

$$\mu^4 = \frac{4\xi^2\sigma^2}{4\sigma\Delta - \Gamma^2} . \tag{23}$$

B. Explicit form of eigenfunction of the Lewis invariant and Lewis phase

The eigenfunction of the invariant operator $I(t)$, $\phi_{n,m-n}(r,\theta)$, for the exponential solution set (as calculated in our earlier publication[4,6]) is given by,

$$\begin{aligned}\phi_{n,m-n}(r,\theta) &= \lambda_n \frac{(i\mu e^{-\Gamma t/2})^m}{\sqrt{m!}} r^{n-m} e^{i(m-n)\theta - \frac{2\sigma + i\mu^2\Gamma}{4\sigma\mu^2 e^{-\Gamma t}}r^2}\\ & U\left(-m, 1-m+n, \frac{r^2 e^{\Gamma t}}{\mu^2}\right) , \end{aligned} \tag{24}$$

where λ_n is given by

$$\lambda_n^2 = \frac{1}{\pi\, n!\, [\mu^2\, exp\,(-\Gamma t)]^{1+n}} . \tag{25}$$

In order to calculate the NC parameters explicitly we need to choose some explicit form of the damping factor i.e. $f(t)$, angular frequency i.e. $\omega(t)$ and the external magnetic field i.e. $B(t)$. Hence we preselect

$$f(t) = e^{-\Gamma t}, \omega(t) = \omega_0, B(t) = B_0\, e^{\Gamma t} sin(\Gamma t + \chi) . \tag{26}$$

Substituting the exponential form of $a(t)$ and $b(t)$ and the above parameters in the first two relations of Eqn.(7), the NC parameters take the form as,

$$\begin{aligned}\theta(t) &= \frac{8MqB_0\, e^{-\Gamma t}}{q^2B_0^2 sin^2(\Gamma t + \chi) + 4M^2\omega_0^2}\\ &\times \left[\sqrt{\frac{\sigma sin^2(\Gamma t + \chi)}{4M} + \frac{\omega_0^2(M\sigma - 1)}{q^2B_0^2}} - \frac{sin(\Gamma t + \chi)}{2M}\right] ;\\ \Omega(t) &= e^{\Gamma t}\left[-qB_0 sin(\Gamma t + \chi) + 2\sqrt{M\Delta - M^2\omega_0^2}\right] . \end{aligned} \tag{27}$$

Substitution of these expressions in the last relation in Eqn.(7) finds the form of $c(t)$ as,

$$c(t) = \frac{1}{q^2B_0^2 sin^2(\Gamma t+\chi)+4M^2\omega_0^2}\left[-2qB_0M\omega_0^2 sin(\Gamma t+\chi) - \frac{q^2B_0^2 sin^2(\Gamma t+\chi)}{M}\sqrt{M\Delta - M^2\omega_0^2}\right.$$
$$\left. + \left(4M^2\omega_0^2 + 2qB_0 sin(\Gamma t+\chi)\sqrt{M\Delta - M^2\omega_0^2}\right)\sqrt{\frac{q^2B_0^2\sigma sin^2(\Gamma t+\chi)}{4M} + \omega_0^2(M\sigma - 1)}\right] + \sqrt{\frac{\Delta}{M} - \omega_0^2}\ . \qquad (28)$$

While the explicit form of $c(t)$ contains only sinusoidal terms , the NC parameters contain both of the exponential and sinusoidal terms.
The exact explicit form of the Lewis phase which can be obtained by substituting the expressions of $a(t)$, $\rho(t)$ and $c(t)$ in Eqn.(21), is shown in the Appendix.

V. PERIODIC NATURE OF ENERGY EXPECTATION VALUE CORRESPONDING TO EXPONENTIAL EP SOLUTION

In order to study the oscillator's behaviour with respect to time , we calculate the expectation value of energy which is already found in our previous study [6] that ,

$$\langle E_{n,m-n}(t)\rangle = \frac{1}{2}(n+m+1)\left[b(t)\rho^2(t) + \frac{a(t)}{\rho^2(t)} + \frac{\dot{\rho}^2(t)}{a(t)}\right] + \frac{(n-m)}{2}c(t); \qquad (29)$$

where $c(t)$ follows the third relation of Eqn.(7). Substituting the Eqn.(22) and Eqn.(26) in the above relation , the explicit form of the energy expectation value is found as,

$$\langle E_{n,-n}(t)\rangle = (n+1)\mu^2\Delta + \frac{n}{q^2B_0^2 sin^2(\Gamma t+\chi)+4M^2\omega_0^2}\left[-2qB_0M\omega_0^2 sin(\Gamma t+\chi) - \frac{q^2B_0^2 sin^2(\Gamma t+\chi)}{M}\sqrt{M\Delta - M^2\omega_0^2}\right.$$
$$\left. + \left(4M^2\omega_0^2 + 2qB_0 sin(\Gamma t+\chi)\sqrt{M\Delta - M^2\omega_0^2}\right)\sqrt{\frac{q^2B_0^2\sigma sin^2(\Gamma t+\chi)}{4M} + \omega_0^2(M\sigma - 1)}\right] + n\sqrt{\frac{\Delta}{M} - \omega_0^2}\ . \qquad (30)$$

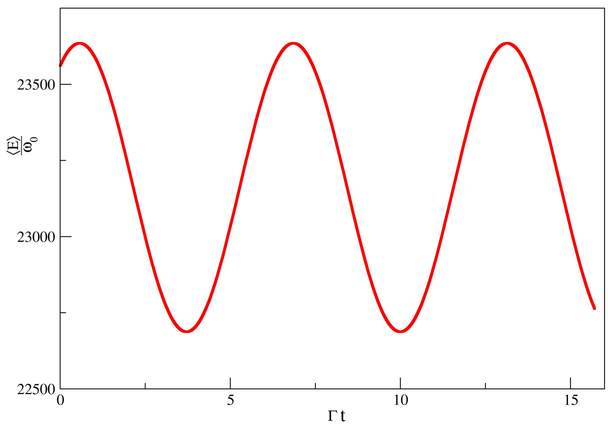

FIG. 1: *A study of the variation of expectation value of energy, scaled by* $\frac{1}{\omega_0}$ *(*$\frac{\langle E\rangle}{\omega_0}$*) in order to make it dimensionless, as we vary* Γt *(again a dimensionless quantity). Here we consider mass M=1, charge q=1, magnetic field* B_0*=*10^2*,* μ*=1,*Δ*=*10^7*,* σ*=*10^7*,* ω_0*=*10^3 *and* Γ*=1 in natural units. The constants n=1 and m=0. The expectation value of energy* $\langle E\rangle$ *is calculated for an exponentially varying EP solution set when* $f(t) = e^{-\Gamma t}$ *,* $\omega(t) = \omega_0$ *and* $B(t) = B_0\, e^{\Gamma t} sin(\Gamma t + \chi)$*.* $\frac{\langle E\rangle}{\omega_0}$ *is found to be a periodic function of time.*

It is very interesting to note that the dynamics of the energy is completely periodic (as seen in the Fig.1) even in the presence of damping. It happens mainly due to the fact that the interplay between the NC parameters and the applied magnetic field makes a perfect balance among the exponentially increasing terms and the exponentially decaying terms. Hence , the terms which remain in the above expression are sinusoidal which are basically periodic function with time.

VI. CONCLUSION

We now summarize our results. Here the model considered in our recent study [6] to study a two-dimensional damped harmonic oscillator affected by an external time varying magnetic field in time dependent noncommutative space is considered again. Then the Hamiltonian is mapped to commutative space by the standard Bopp-shift relations. The exact solution of the model is obtained by using the well known method of Lewis invariant. The time dependent hermitian invariant whose eigenfunction relates the eigenfunction of the original Hamiltonian, after being combined with a time dependent phase factor, known as Lewis phase in literature, is associated with a non-linear differential equation known as the Ermakov-Pinney (EP) equation. In order to obtain the explicit form of the solution of Hamiltonian, an explicit solution set of EP equation is considered in terms of exponential function with respect to time and the explicit form of the time dependent parameters damping factor, angular frequency and the external magnetic field are also chosen in terms of exponential and periodic functions with respect to time. Interestingly, periodicity of the magnetic field is reflected in the energetics of the system as seen from the time evolution of the corresponding energy expectation value.

VII. APPENDIX

The time dependent Lewis phase (as mentioned in section IV-B) which is necessary to produce the eigenfunction of the model is as follows,

$$
\begin{aligned}
\Theta_{n,l}(t) = & \frac{(n+l)\omega_0}{\Gamma\sqrt{M\sigma-1}}\left[M\sigma\left\{EllipticF\left(\Gamma t+\chi,-\frac{q^2B_0^2\sigma}{4M\omega_0^2(M\sigma-1)}\right)-EllipticF\left(\chi,-\frac{q^2B_0^2\sigma}{4M\omega_0^2(M\sigma-1)}\right)\right\}\right.\\
& \left.-\left\{EllipticPi\left(-\frac{q^2B_0^2}{4M^2\omega_0^2},\Gamma t+\chi,-\frac{q^2B_0^2\sigma}{4M\omega_0^2(M\sigma-1)}\right)-EllipticPi\left(-\frac{q^2B_0^2}{4M^2\omega_0^2},\chi,-\frac{q^2B_0^2\sigma}{4M\omega_0^2(M\sigma-1)}\right)\right\}\right]\\
& +\frac{(n+l)\sqrt{M\Delta-M^2\omega_0^2}}{\Gamma}\left[\frac{2\omega_0}{\sqrt{q^2B_0^2+4M^2\omega_0^2}}\left\{tan^{-1}\frac{\sqrt{2}\omega_0qB_0cos(\Gamma t+\chi)}{\sqrt{q^2B_0^2+4M^2\omega_0^2}\sqrt{2\omega_0^2(M\sigma-1)+\frac{q^2B_0^2\sigma}{4M}(1-cos[2(\Gamma t+\chi)])}}\right.\right.\\
& \left.-tan^{-1}\frac{\sqrt{2}\omega_0qB_0cos\,\chi}{\sqrt{q^2B_0^2+4M^2\omega_0^2}\sqrt{2\omega_0^2(M\sigma-1)+\frac{q^2B_0^2\sigma}{4M}(1-cos[2\chi])}}\right\}\\
& \left.+\frac{i\sqrt{\sigma}}{\sqrt{M}}log\frac{iqB_0\sqrt{\frac{\sigma}{2M}}cos(\Gamma t+\chi)+\sqrt{2\omega_0^2(M\sigma-1)+\frac{q^2B_0^2\sigma}{4M}(1-cos[2(\Gamma t+\chi)])}}{iqB_0\sqrt{\frac{\sigma}{2M}}cos(\chi)+\sqrt{2\omega_0^2(M\sigma-1)+\frac{q^2B_0^2\sigma}{4M}(1-cos[2(\chi)])}}\right]+(n+l)\left[\sqrt{\frac{\Delta}{M}-\omega_0^2}-\frac{\sigma}{\mu^2}\right]t\\
& +\frac{(n+l)2M\omega_0^2}{\Gamma\sqrt{q^2B_0^2+4M^2\omega_0^2}}\left(tanh^{-1}\frac{qB_0cos(\Gamma t+\chi)}{\sqrt{q^2B_0^2+4M^2\omega_0^2}}-tanh^{-1}\frac{qB_0cos(\chi)}{\sqrt{q^2B_0^2+4M^2\omega_0^2}}\right)-\frac{(n+l)\sqrt{M\Delta-M^2\omega_0^2}\,t}{M}\\
& +\frac{(n+l)2\omega_0\sqrt{M\Delta-M^2\omega_0^2}}{\Gamma\sqrt{q^2B_0^2+4M^2\omega_0^2}}\left[tan^{-1}\left(\frac{\sqrt{q^2B_0^2+4M^2\omega_0^2}}{2M\omega_0}tan(\Gamma t+\chi)\right)-tan^{-1}\left(\frac{\sqrt{q^2B_0^2+4M^2\omega_0^2}}{2M\omega_0}tan(\chi)\right)\right].
\end{aligned}
\tag{31}
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