A study of an Exotic Quantum Damped Harmonic Oscillator

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We have studied a two-dimensional damped harmonic oscillator in time dependent noncommutative space in an earlier communication [Int. J. Theor. Phys. **59**, 3852 (2020)]. In this paper we have reconsidered the same model and extended the discussion by introducing a generalised solution of Ermakov-Pinney (EP) equation. We proceed to estimate the expectation value of energy in this state.

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I. INTRODUCTION

Researchers have done quantum mechanical analysis of damped harmonic oscillators extensively. In an earlier communication we had studied the quantum damped harmonic oscillator in noncommutative [NC] space[12]. By specifying NC space we indicate a space in which we have additional NC commutation relations between the position and momentum canonical coordinates of the form, $[X_1, X_2] = i\theta$ and $[P_1, P_2] = i\Omega$, where θ and Ω are positive real constants. The study of such systems in NC space is believed to be essential in the exciting fields of loop quantum gravity[4]and string theory[2,3].

In our previous communication we had set up the hamiltonian for quantum damped harmonic oscillator in two-dimensional NC space. We find exact solutions of this Hamiltonian for various choices of parameters of the problem. Various choices of these parameters lead to rationally decaying, exponentially decaying and elementary solutions. We have also proceeded to estimate the expectation values of energy for these solutions. It is seen that there is a certain time range within which the energy remains real. This time range is actually imposed by the condition that the Hamiltonian has to be hermitian.

In the present communication we intend to extend our study to generate some generic solutions of our model quantum mechanical system in NC space. For a particular choice of parameters we get an exact closed form corresponding to this generic solution. For this exact closed form solution we also calculate the expectation value of energy. We observe that there is a certain time range within which the energy is found to be real.

In our study, we first revisit our earlier Hamiltonian in two-dimensional NC space. We also navigate through the mechanism required to solve the Hamiltonian in Section 2. In the next Section we generate some generic forms of solutions of the Hamiltonian. For certain choice of parameters we get an exact solution of the Hamiltonian corresponding to the generic form. In Section 4 we calculate the energy expectation value. Finally we summarize our results in Section 5.

II. REVISITING THE TWO-DIMENSIONAL HARMONIC OSCILLATOR IN NONCOMMUTATIVE SPACE

Our Hamiltonian for the damped harmonic oscillator in 2 dimensional non commutative space has the following form,

$$H(t) = \frac{f(t)}{2M}({P_1}^2 + {P_2}^2) + \frac{M\omega^2(t)}{2f(t)}({X_1}^2 + {X_2}^2) \quad (1)$$

where the damping resistance $f(t)$ has the following form,

$$f(t) = e^{-\int_0^t \eta(s)ds}. \quad (2)$$

Here $\eta(s)$ is the coefficient of damping. We consider $\omega(t)$ to be the oscillator's angular frequency which is time-dependent and M is the mass of the oscillator. The Hamiltonian is expressed in terms of NC coordinates (X_i, P_i) which are related to the corresponding commutative space variables (x_i, p_i) by the standard Bopp-shift relations[8] which are of the following form (considering $\hbar = 1$):

$$X_1 = x_1 - \frac{\theta(t)}{2}p_2 \ , \ X_2 = x_2 + \frac{\theta(t)}{2}p_1 \ , \quad (3)$$

$$P_1 = p_1 + \frac{\Omega(t)}{2}x_2 \ , \ P_2 = p_2 - \frac{\Omega(t)}{2}x_1 \ . \quad (4)$$

The Hamiltonian in terms of (x_i, p_i) coordinates is therefore given by the following relation,

$$H = \frac{a(t)}{2}({p_1}^2+{p_2}^2)+\frac{b(t)}{2}({x_1}^2+{x_2}^2)+c(t)(p_1x_2-p_2x_1) \ . \quad (5)$$

The time dependent coefficients in the above Hamiltonian are given as,

$$a(t) = \frac{f(t)}{M} + \frac{M\omega^2(t)\theta^2(t)}{4f(t)} \ , \ b(t) = \frac{f(t)\Omega^2(t)}{4M} + \frac{M\omega^2(t)}{f(t)} \ ,$$
$$c(t) = \frac{1}{2}\left[\frac{f(t)\Omega(t)}{M} + \frac{M\omega^2(t)\theta(t)}{f(t)}\right]. \tag{6}$$

A. Solving the Hamiltonian

In order to find the solutions of the model Hamiltonian we follow the route suggested by Lewis *et.al.*[1] in their work. So we begin by setting up the time-dependent Hermitian invariant operator $I(t)$ for the Hamiltonian $H(t)$ (given by Eqn.(5)). This is because if one can solve for the eigenfunctions of $I(t)$, $\phi(x_1, x_2)$, such that,

$$I(t)\phi(x_1, x_2) = \epsilon\phi(x_1, x_2) \tag{7}$$

where ϵ is an eigenvalue of $I(t)$ corresponding to eigenstate $\phi(x_1, x_2)$, one can obtain the eigenstates of $H(t)$, $\psi(x_1, x_2, t)$, using the relation given by Lewis *et. al.*[1] which is as follows,

$$\psi(x_1, x_2, t) = e^{i\Theta(t)}\phi(x_1, x_2). \tag{8}$$

Here the function $\Theta(t)$ which is actually the phase factor is real.

B. The Time Dependent Invariant

Next, following the approach taken by Lewis *et.al.*[1], we need to construct the operator $I(t)$ which is an invariant with respect to time, corresponding to the Hamiltonian $H(t)$, as mentioned earlier, such that $I(t)$ satisfies the condition,

$$\frac{dI}{dt} = \partial_t I + \frac{1}{i}[I, H] = 0. \tag{9}$$

The procedure is to choose the Hermitian invariant $I(t)$ to be of the same homogeneous quadratic form defined by Lewis *et. al.*[1] for time-dependent harmonic oscillators. However, since we are dealing with a two-dimensional system in the present study, $I(t)$ takes on the following form,

$$I(t) = \alpha(t)({p_1}^2 + {p_2}^2) + \beta(t)({x_1}^2 + {x_2}^2) + \gamma(t)(x_1 p_1 + p_2 x_2). \tag{10}$$

Here we will consider $\hbar = 1$ since we choose to work in natural units. Now, using the form of $I(t)$ defined by Eqn.(10) in Eqn.(9) and equating the coefficients of the canonical variables, we get the following relations,

$$\dot{\alpha}(t) = -a(t)\gamma(t) \ , \ \dot{\beta}(t) = b(t)\gamma(t) \ ,$$
$$\dot{\gamma}(t) = 2\left[b(t)\alpha(t) - \beta(t)a(t)\right] \tag{11}$$

where dot denotes derivative with respect to time t.

To express the above three time dependent parameters α,β and γ in terms of a single time dependent parameter, we parametrize $\alpha(t) = \rho^2(t)$. Substituting this in the first and third relation in the Eqn.(11), we get the other two parameters in terms of $\rho(t)$ as,

$$\gamma(t) = -\frac{2\rho\dot{\rho}}{a(t)} \ , \ \beta(t) = \frac{1}{a(t)}\left[\frac{\dot{\rho}^2}{a(t)} + \rho^2 b + \frac{\rho\ddot{\rho}}{a(t)} - \frac{\rho\dot{\rho}\dot{a}}{a^2}\right] \tag{12}$$

Now, substituting the value of β in the second relation in Eqn.(11), we get a non-linear equation in $\rho(t)$ which has the form of the non-linear Ermakov-Pinney (EP) equation with a dissipative term[5–7]. The form of the non-linear equation is as follows,

$$\ddot{\rho} - \frac{\dot{a}}{a}\dot{\rho} + ab\rho = \xi^2\frac{a^2}{\rho^3} \ . \tag{13}$$

where ξ^2 is a constant of integration. This equation has similar form to the EP equation obtained in[5], which is expected since our $H(t)$ has the same form as theirs. However, once again we should recall the fact that the explicit form of the time-dependent coefficients are different due to the presence of damping.
Now, using the EP equation we get a simpler form of β as,

$$\beta(t) = \frac{1}{a(t)}\left[\frac{\dot{\rho}^2}{a(t)} + \frac{\xi^2 a(t)}{\rho^2}\right]. \tag{14}$$

Next, substituting the expressions of α, β and γ in Eqn.(10), we get the following expression for $I(t)$,

$$I(t) = \rho^2({p_1}^2 + {p_2}^2) + \left(\frac{\dot{\rho}^2}{a^2} + \frac{\xi^2}{\rho^2}\right)({x_1}^2 + {x_2}^2) - \frac{2\rho\dot{\rho}}{a}(x_1 p_1 + p_2 x_2). \tag{15}$$

The form of the Lewis invariant in polar coordinate system is as follows, **(including** $\hbar$)

$$I(t) = \frac{\xi^2}{\rho^2}r^2 + \left(\rho p_r - \frac{\dot{\rho}}{a}r\right)^2 + \left(\frac{\rho p_\theta}{r}\right)^2 - \left(\frac{\rho\hbar}{2r}\right)^2 \tag{16}$$

The solution of the EP equation under various physically significant conditions shall be discussed later.
Using this form of the invariant the form of the corresponding solution of the Hamiltonian (as seen from our previous study[12]) is given by,

$$\begin{aligned}
\psi_{n,m-n}(r,\theta,t) &= e^{i\Theta_{n,m-n}(t)}\phi_{n,m-n}(r,\theta) \\
&= \lambda_n \frac{(i\sqrt{\hbar}\rho)^m}{\sqrt{m!}} \exp\left[im\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right)dT\right] \\
&\quad \times r^{n-m} e^{i(m-n)\theta - \frac{a(t) - i\rho\dot{\rho}}{2a(t)\hbar\rho^2}r^2} \\
&\quad \times U\left(-m, 1-m+n, \frac{r^2}{\hbar\rho^2}\right).
\end{aligned} \tag{17}$$

where λ_n is given by

$$\lambda_n^2 = \frac{1}{\pi n!(\hbar\rho^2)^{1+n}} \ . \tag{18}$$

Here, $U\left(-m, 1-m+n, \frac{r^2}{\hbar\rho^2}\right)$ is Tricomi's confluent hypergeometric function[9,10].

Where following the study by Dey *et al*[5] the phase factor $\Theta(t)$ has the following form,

$$\Theta_{n,l}(t) = (n+l)\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right) dT = m\int_0^t \left(c(T) - \frac{a(T)}{\rho^2(T)}\right) dT\ . \tag{19}$$

Using this form we are going to calculate the phase as a function of time for a generic solution of the Hamiltonian.**The constant ξ^2 in the eqn.(13) is set as 1 while producing the above solution of the hamiltonian.**

III. GENERIC SOLUTIONS FOR THE NONCOMMUTATIVE DAMPED OSCILLATOR

Here we intend to define the Hamiltonian parameters in terms of $\rho(t)$, which is a time dependent parameter of Ermakov-Pinney equation. Such parametrization provides us a generic solution set as discussed in details below.

A. The solution set

A simple method of obtaining a solution of the EP equation is provided in[12]. We first choose $\rho(t)$ to be any arbitrary time dependent function. The time derivative of $\rho(t)$ is proportional to $a(t)$, that is, $a(t) = k_1 \times \dot{\rho}(t)$, where k_1 is an arbitrary constant. Next we set the parameter $b(t) = k_2 \times \frac{a(t)}{\rho^4(t)}$, where k_2 is also another arbitrary constant. We observe that these parameters $a(t)$ and $b(t)$ always satisfy the EP equation along with a certain constraint condition amongst the arbitrary constants. As the Hamiltonian parameters are not given any explicit functional form, rather they are expressed in terms of the parameter $\rho(t)$, we call this solution set as the generic solution which has the following form,

$$\rho = \rho(t) \quad , \quad a(t) = k_1\dot{\rho}(t) \quad , \quad b(t) = k_1 k_2 \frac{\dot{\rho}(t)}{\rho^4(t)}\ ; \tag{20}$$

where k_1, k_2 are arbitrary constants. It is very easy to check that the above set of parameters would always satisfy the EP equation along with the following constraint relation,

$$k_2 = \xi^2 \tag{21}$$

B. Study of the corresponding eigenfunction

The eigenfunction of the invariant operator $I(t)$, $\phi_{n,m-n}(r,\theta)$, for this generic solution set (as deciphered from our earlier publication[12]) is given by,

$$\phi_{n,m-n}(r,\theta) = \langle r,\theta|n, m-n\rangle \tag{22}$$

$$= \lambda_n \frac{(i\sqrt{\hbar}\rho)^m}{\sqrt{m!}} r^{n-m} e^{i(m-n)\theta - \frac{k_1 - i\rho}{2k_1\hbar\rho^2}r^2} \times U\left(-m, 1-m+n, \frac{r^2}{\hbar\rho^2}\right) \tag{23}$$

where λ_n is given by

$$\lambda_n^2 = \frac{1}{\pi n!(\hbar\rho^2)^{1+n}}\ . \tag{24}$$

We now apply the generic form of $a(t)$ and $b(t)$ in the first two relations of Eqn.(6),and we have the explicit form of the time dependent NC parameters in terms of $\rho(t)$ as,

$$\theta(t) = \frac{2f(t)}{M\omega(t)}\sqrt{\frac{Mk_1\dot{\rho}(t)}{f(t)} - 1}\ ;$$

$$\Omega(t) = \frac{2}{f(t)}\sqrt{\frac{Mk_1k_2\dot{\rho}(t)f(t)}{\rho^4(t)} - M^2\omega^2(t)} \tag{25}$$

Substituting these expressions in the expression for $c(t)$ in Eqn.(6), we get the following relation for $c(t)$,

$$c(t) = \sqrt{\frac{k_1k_2\dot{\rho}(t)f(t)}{M\rho^4(t)} - \omega^2(t)} + \omega(t)\sqrt{\frac{Mk_1\dot{\rho}(t)}{f(t)} - 1} \tag{26}$$

Substituting the expressions of $a(t)$, $\rho(t)$ and $c(t)$ in Eqn.(19), we can express the phase factor in terms of ρ as ,

$$\Theta_{n,l}(t) = (n+l)\int_0^t \left(c(t) - \frac{k_1\dot{\rho}}{\rho^2}\right) dT \tag{27}$$

$$= (n+l)\left[k_1\left\{\rho^{-1}(t')|_{t'=t} - \rho^{-1}(t')|_{t'=0}\right\} + \int_0^t c(t)\, dT\right] \tag{28}$$

where the form of $c(t)$ has explicit dependence on ρ according to the eqn. (26).

C. Generic solution set of EP equation : A special case

Inorder to ensure that the Hamiltonian parameters are real, we choose the variable $\rho(t)$ to be an exponentially increasing function of time, as follows,

$$\rho(t) = \mu\, e^{\Gamma\, t} \,. \tag{29}$$

The above choice would immediately set the forms of the two parameters $a(t)$ and $b(t)$ of the Hamiltonian as,

$$a(t) = k_1\mu\Gamma\, e^{\Gamma\, t}\,,\; b(t) = \frac{k_1 k_2 \Gamma}{\mu^3}\, e^{-3\Gamma\, t}\,. \tag{30}$$

It is relevant to mention that the exponential EP solution obtained in[5] and used in[12] are slightly different from the above solution because those do not hold the relation showed in eqn.(20). In order to obtain an exact solution, we choose both the functions $\omega(t)$ and $f(t)$ as follows,

$$f(t) = 1\;,\; \omega(t) = \omega_0 e^{-\Gamma\, t/2}\,. \tag{31}$$

Substituting the forms of $\rho(t), f(t)$ and $\omega(t)$ in the eqn.(25),we get the forms of the time dependent NC parameters as,

$$\begin{aligned}\theta(t) &= \frac{2}{M\omega_0} e^{\Gamma\, t/2}\sqrt{Mk_1\mu\Gamma e^{\Gamma\, t} - 1}\,,\\ \Omega(t) &= 2\sqrt{\frac{Mk_1k_2\Gamma}{\mu^3}e^{-3\Gamma\, t} - M^2\omega_0^2 e^{-\Gamma\, t}}\,.\end{aligned} \tag{32}$$

Substituting the forms of $\rho(t), f(t)$ and $\omega(t)$ in the eqn.(26),we get the form of the Hamiltonian parameter $c(t)$ as,

$$c(t) = \omega_0\sqrt{Mk_1\mu\Gamma - e^{-\Gamma\, t}} + \sqrt{\frac{k_1k_2\Gamma}{M\mu^3}e^{-3\Gamma\, t} - \omega_0^2 e^{-\Gamma\, t}}\,. \tag{33}$$

Finally we have an exact form of the phase factor from the eqn.(28) as

$$\begin{aligned}\Theta_{n,l}(t) = &\frac{(n+l)k_1}{\mu}(e^{-\Gamma\, t} - 1) + \frac{(n+l)\,\omega_0}{\Gamma}\Big[2\left(\sqrt{Mk_1\mu\Gamma - 1} - \sqrt{Mk_1\mu\Gamma - e^{-\Gamma\, t}}\right) + \Gamma\, t\sqrt{Mk_1\mu\Gamma}\\ &+2\sqrt{Mk_1\mu\Gamma}\; log\frac{\sqrt{Mk_1\mu\Gamma} + \sqrt{Mk_1\mu\Gamma - e^{-\Gamma\, t}}}{\sqrt{Mk_1\mu\Gamma} + \sqrt{Mk_1\mu\Gamma - 1}}\Big] + \frac{2(n+l)}{3\Gamma}\sqrt{\frac{k_1k_2\Gamma}{M\mu^3}}\Big[2F1\left(-\frac{3}{4}, -\frac{1}{2}, \frac{1}{4}, \frac{\omega_0^2\mu^3 M}{k_1k_2\Gamma}\right)\\ &- e^{-3\Gamma t/2} 2F1\left(-\frac{3}{4}, -\frac{1}{2}, \frac{1}{4}, \frac{\omega_0^2\mu^3 M}{k_1k_2\Gamma}e^{2\Gamma\, t}\right)\Big]\;;\end{aligned} \tag{34}$$

Here, $2F1(a, b, c, z)$[13] is known as Gauss hypergeometric function[11].

IV. ANALYSIS OF THE EXPECTATION VALUE OF ENERGY FOR GENERIC SOLUTION

In this section, we intend to calculate the expectation value of energy. It is shown in[12] that the expectation value of energy $\langle E_{n,m-n}(t)\rangle$ with respect to energy eigenstate $\psi_{n,m-n}(r,\theta,t)$ can be expressed as,

$$\begin{aligned}\langle E_{n,m-n}(t)\rangle = &\frac{1}{2}(n+m+1)\left[b(t)\rho^2(t) + \frac{a(t)}{\rho^2(t)} + \frac{\dot{\rho}^2(t)}{a(t)}\right]\\ + &c(t)\,(n-m)\end{aligned} \tag{35}$$

The generic form of the EP solution in eqn.(20) provides the generic form of the expectation value of energy in terms of ρ as

$$\langle E_{n,-n}(t)\rangle = \frac{(n+1)}{2}\left[k_1k_2\frac{\dot{\rho}}{\rho^2} + k_1\frac{\dot{\rho}}{\rho^2} + \frac{\dot{\rho}}{k_1}\right] + n\left[\sqrt{\frac{k_1k_2\dot{\rho}(t)f(t)}{M\rho^4(t)}} - \omega^2(t) + \omega(t)\sqrt{\frac{Mk_1\dot{\rho}(t)}{f(t)}} - 1\right]\,. \tag{36}$$

For the case where we consider $\rho(t) = \mu\, e^{\Gamma\, t}$ in the eqn.(29), $f(t) = 1$ and $\omega(t) = \omega_0\, e^{-\Gamma\, t/2}$ in the eqn.(31),the expectation value of energy for the ground state has the following expression,

$$\langle E_{n,-n}(t)\rangle = \frac{k_1(n+1)\Gamma}{2\mu}\left[e^{-\Gamma t}(k_2+1)+\frac{\mu^2}{k_1^2}e^{\Gamma t}\right]+n\left[\omega_0\sqrt{Mk_1\mu\Gamma - e^{-\Gamma t}}+\sqrt{\frac{k_1k_2\Gamma}{M\mu^3}e^{-3\Gamma t}-\omega_0^2e^{-\Gamma t}}\right] . \quad (37)$$

The deduced expression of energy in eqn.(37) provides a range of time within which the ground state energy is real,

$$\frac{1}{\Gamma}\, ln\left(\frac{1}{Mk_1\mu\Gamma}\right) \leq t \leq \frac{1}{2\Gamma}\, ln\left(\frac{k_1k_2\Gamma}{M\mu^3\omega_0^2}\right) . \quad (38)$$

So, only within this time range the Hamiltonian is hermitian. The first condition arises from the fact that the NC parameter $\theta(t)$ has to be real and the second condition arises similarly from the fact that the NC parameter $\Omega(t)$ has to be real. These conditions also ensure that the energy does not diverge with increasing time.

V. CONCLUSION

We now summarize the results obtained from this study. In this paper we have used the model considered in[12] to study a two-dimensional damped harmonic oscillator in time dependent noncommutative space. We map the Hamiltonian to commutative space by using a shift of variables connecting the noncommutative and commutative space, known in literature as Bopp-shift. We have then obtained the exact solution of this time dependent system by using the well known Lewis invariant approach which in turn leads to a non-linear differential equation known as the Ermakov-Pinney (EP) equation. We consider a generic solution where we assume the explicit form of a single Hamiltonian parameter. The other two Hamiltonian parameters are defined in terms of this Hamiltonian parameter. We also proceed to calculate the expectation value of the Hamiltonian. Expectedly, the expectation value of energy varies with time. For the generic solution of EP equation two different noncommutative parameters set the lower and upper bounds within which the energy expectation value is real.

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[13] Here a, b, c should not be confused with the hamiltonian parameters.